Exhaustive search of convex pentagons which tile the plane

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Abstract

We present an exhaustive search of all families of convex pentagons which tile the plane. This research shows that there are no more than the already 15 known families. In particular, this implies that there is no convex polygon which allows only non-periodic tilings.

1 Introduction

If one asks which convex polygon can tile the plane (allowing translations, rotations and mirrors), the case of pentagons is the only opened case: every triangle and quadrilateral tiles the plane, there are 3 families of hexagons which tile the plane, and no convex polygon with more than six sides can tile the plane (see for example [4]).

The research of families of pentagons which tile the plane has an intriguing history. The first families of pentagons were presented by Reinhardt in 1918 [3]. Kershner presented new families, and announced that the list was complete in 1968 [1]. But news families were found afterwards, one by R. James in 1975, three by an amateur mathematician M. Rice in 1977, and one by R. Stein in 1985. Finally, the fifteenth (and last) family was found by Mann, McLoud and Von Derau in 2015 (see [2]).

We present here an exhaustive search of all families of pentagons which tile the plane. This search is not restricted to periodic tilings, and does not find any new family. The key point is that there are only finitely many, 371, families of angle conditions to consider.

In Section 2, we introduce the notations, and we show that if a pentagon tiles the plane, then there is a tiling such that every vertex type has positive density. In Section 3, we show that there are only finitely possible sets of vertex types in a positive density tiling by a pentagon. Then, in Section 4, we present a backtracking technique to search a tiling, when we fix the set of vertex types. This backtracking algorithm does not find any new family of pentagons.

2 Positive density tilings

Throughout this section, we fix convex pentagon $\mathcal{P} \subset \mathbb{R}^2$. Let $s_1, \dots s_5 \in \mathbb{R}^2$ be its 5 vertices in clockwise order. For $i \in \{1, ..5\}$, let $\alpha_i \times \pi$ be the angle at vertex s_i and let $\alpha = (\alpha_1, \dots, \alpha_5)$. We recall that $\sum_{i=1}^{5} \alpha_i = 3$.

A *tiling* $\mathcal{T}$ of $\mathcal{S}$ by $\mathcal{P}$ is a set of subset of $\mathcal{S}$ such that:

- $\cup_{P \in \mathcal{T}} P = \mathcal{S}$,
- for every $P \in \mathcal{T}$, there is an isometry of the plane h_P such that $h_P(\mathcal{P}) = P$,
- for every $P, Q \in \mathcal{T}$ with $P \neq Q$, $\mathring{P} \cap \mathring{Q} = \emptyset$ (where $\mathring{P}$ is the interior of P with the usual topology on $\mathbb{R}^2$).

Given a tiling of $\mathcal{S}$, we fix an isometry h_P for every $P \in \mathcal{T}$ such that $P = h_P(\mathcal{P})$. Elements of $\mathcal{T}$ are called a *tiles*. A point s in $\mathcal{S}$ is a vertex of $\mathcal{T}$ if it is a vertex of at least one tile in $\mathcal{T}$ (that is, there is a $P \in \mathcal{T}$ and a $i \in \{1, \dots, 5\}$ such that $s = h_P(s_i)$. The set of vertices of $\mathcal{T}$ is denoted $\mathcal{V}(\mathcal{T})$. A *tiling of the plane* is a tiling of $\mathbb{R}^2$.

From now on, we fix a tiling $\mathcal{T}$ of the plane by $\mathcal{P}$. Let $P, Q \in \mathcal{T}$ and $s \in \mathcal{V}(\mathcal{T})$. We say that Q *follows* P *around* s if there are i and j such that $h_P(s_i) = h_Q(s_j) = s$, $P \cap Q \neq s$, and Q are just after P if we turn around s in clockwise order.

We distinguish two types of vertices: full and half. Let s be a vertex of a tilling $\mathcal{T}$.

If there is a sequence $\mathfrak{s}_s = (P_1, \dots, P_k)$ of tiles such that for every $i \in \{1, \dots k\}$, P_{i+1} follows P_i around s, where the indices are taken modulo k, we say that s is *full*. Otherwise, we say that s is *half*: there is a tile around s for which s is not a vertex, but is on the border. Then there is a maximal sequence $\mathfrak{s}_s = (P_1, \dots, P_k)$ of tiles such that for every $i \in \{1, \dots k-1\}$, P_{i+1} follows P_i around s. (There is no tile Q such that Q follows P_k, or P_1 follows Q around s.)

The *vector type* of s, denoted $V(s)$, is $(c_1, \dots, c_5) \in \mathbb{N}^5$, where c_i is the number of tiles P in $\mathfrak{s}_s$ such that $h_P(s_i) = s$. The *corrected vector type* of s, denoted $V^c(s)$, is either $V(s)$ if s is full, or $2 \times V(s)$ if s is half. We have in any case $V^c(s) \cdot \alpha = 2$.

Let $G = (\mathcal{T}, \mathcal{A})$ be the oriented graph, called the *underliyng graph* of $\mathcal{T}$, such that there is an arc between P and Q if P follows Q around s for a vertex s in $\mathcal{T}$. Moreover, we label each tile P in the graph by "+" if h_P is a translation or a rotation, or by "-" if h_P is a glide reflexion. We label each arc (P, Q) by (i, j) with $h_P(s_i) = h_Q(s_j) = s$, where s is such that P follows Q around s.

Let $\mathcal{T}' \subset \mathcal{T}$. The subgraph induced by $\mathcal{T}'$ is the graph $G[\mathcal{T}'] = (\mathcal{T}', \mathcal{A}')$, where $\mathcal{A}' = \mathcal{A} \cap \mathcal{T}^2$. A tile P in $\mathcal{T}'$ is a *frontier tile* if there is a $Q \in \mathcal{T} \setminus \mathcal{T}'$ such that either $(P, Q) \in \mathcal{A}$ or $(Q, P) \in \mathcal{A}$. The set of frontier tiles of H is denoted $\mathcal{FT}(H)$.

Two graphs are *isomorphic* if there exists a bijection that preserves labels.

Given an induced subgraph H of G, we denote by $\mathcal{S}(H) = \bigcup_{P \in \mathcal{T}(H)} P$, where $\mathcal{T}(H)$ is the set of tiles in H. (Note that $\mathcal{T}(H)$ is then a tiling of $\mathcal{S}(H)$.)

The set of vertices of H is denoted $\mathcal{V}(H) = \mathcal{V}(\mathcal{T}(H))$. A vertex s is an *interior vertex* of a subgraph H of G if for every $P \in \mathcal{T}$ such that s is a vertex of P, then $P \in \mathcal{T}(G)$. The set of interior vertices of H is denoted $\mathcal{IV}(H)$. Moreover, the set of interior and half (resp. full) vertices of H is denoted $\mathcal{IV}_H(H)$ (resp. $\mathcal{IV}_F(H)$). (Note: an interior half vertex of H can be on the boundary of $\mathcal{S}(H)$.)

Let $\mathcal{W}(\mathcal{T}) = \{V(s) : s \in \mathcal{V}(\mathcal{T}\})$ and $\mathcal{W}^c(\mathcal{T}) = \{V^c(s) : s \in \mathcal{V}(\mathcal{T})\}$. If $\mathcal{T}$ is clear in the context, we may write $\mathcal{W}$ or $\mathcal{W}^c$. Note that $\mathcal{W}(\mathcal{T})$ and $\mathcal{W}^c(\mathcal{T})$ are finite.

Let $o \in \mathbb{R}^2$ and $r \in \mathbb{R}^+$. Let $\mathcal{T}_{o,r}$ be the set of tiles $P \in \mathcal{T}$ such that $P \cap B(o, r) \neq \emptyset$, where $B(o, r)$ is the disk of radius r and center o. Let $G_{o,r} = G[\mathcal{T}_{o,r}]$ be the graph induced by $\mathcal{T}_{o,r}$.

Proposition 1. *There are constants c and c' in $\mathbb{R}^+$ such that for every $r \in \mathbb{R}^+$, $|\mathcal{FT}(G_{o,r})| \leq c \times r + c'$.*

Proof. This follows from the fact than for every $r \in \mathbb{R}^+$, there is a $n_r \in \mathbb{N}$ such that for every $o \in \mathbb{R}^2$, $|\mathcal{T}_{o,r}| \leq n_r$. □

For $v \in \mathcal{W}(\mathcal{T})$, $o \in \mathbb{R}^2$ and $r \in \mathbb{R}^+$, let:

$$f_{o,r}(v) = \frac{|\{s \in \mathcal{IV}(G_{o,r}) : V(s) = v\}|}{|\mathcal{T}_{o,r}|}.$$

We say that the tiling $\mathcal{T}$ has *positive density* if for every $v \in \mathcal{W}(\mathcal{T})$ and $o \in \mathbb{R}^2$ we have $\liminf_{r \to \infty} f_{o,r}(v) > 0$. (Note that if it is true for one $o \in \mathbb{R}^2$, then it is true for every $o \in \mathbb{R}^2$.)

Lemma 1. *If* $\liminf_{r\to\infty} f_{o,r}(v) = 0$*, then there is a tiling* $\mathcal{T}'$ *of the plane by* $\mathcal{P}$ *such that* $\mathcal{W}(\mathcal{T}') \subseteq \mathcal{W}(\mathcal{T}) \setminus \{v\}$*.*

Proof. Let $d \in \mathbb{R}^+$. We divide $\mathbb{R}^2$ into a grid of $d \times d$ squares $S_{(i,j)}$, with $(i,j) \in \mathbb{Z}^2$. Then we decompose $\mathcal{T}$ into a disjoint union of sets of tiles $\mathcal{T}_{(i,j)}$ such that a tile P is in $\mathcal{T}_{(i,j)}$ if $P \cap S_{(i,j)} \neq \emptyset$ (if there are several possible choices for a tile, we chose arbitrarily). If for every $(i,j) \in \mathbb{Z}^2$, $\mathcal{T}_{(i,j)}$ has one vertex s with $V(s) = v$, then $\liminf_{r\to\infty} f_{o,r}(v) > 0$. Thus, there is a $(i,j) \in \mathbb{Z}^2$ such that $\mathcal{T}_{(i,j)}$ has no vertex s with $V(s) = v$.

For every $d \in \mathbb{R}^+$, there is a subgraph H_d such that $\mathcal{S}(H_d)$ contains a $d \times d$ square, and $\mathcal{W}(H_d) \subseteq \mathcal{W}(\mathcal{T}) \setminus \{v\}$. We keep a connected component H'_d of H_d such that $\mathcal{S}(H'_d)$ contains the center of the square. Then one can construct by compactness an infinite graph G' in which every vertex is an interior vertex, and with $\mathcal{W}(G') \subseteq \mathcal{W}(\mathcal{T}) \setminus \{v\}$. There are three cases: either G' corresponds to a tiling of the plane, or of a half-plane, or of a stripe. In all cases, one can construct a tiling of the plane without vertex of vector type v, and no new vector type. □

Good subsets. We say that a subset $\mathcal{X} \subseteq \mathbb{N}^5$ is *good* if for every $u \in \mathbb{R}^5$ with $\sum u = 0$, either $u \cdot v = 0$ for every $v \in \mathcal{X}$, or there are $v, v' \in \mathcal{X}$ such that $u \cdot v < 0 < u \cdot v'$.

Suppose that $\mathcal{T}$ is a tiling by the pentagon $\mathcal{P}$. By Lemma 1, we assume that $\mathcal{T}$ has positive density.

Proposition 2. *Let* $u \in \mathbb{R}^5$ *such that* $\sum_{i=1}^{5} u_i = 0$*, and there is a* $v^+ \in \mathcal{W}^c$ *with* $u \cdot v^+ > 0$*. Then there is a* $v^- \in \mathcal{W}^c$ *such that* $u \cdot v^- < 0$*.*

Proof. For every $i \in \{1, \dots 5\}$, if we count the number of angles i in $\mathcal{T}(G_{o,r})$, we have:

$$0 \leq |\mathcal{T}(G_{o,r})| - \sum_{v\in\mathcal{W}} v[i] \times |\{s \in \mathcal{IV}(G_{o,r}) : V(s) = v\}| \leq |\mathcal{FT}(G_{o,r})|.$$

Thus, by Proposition 1:

$$\lim_{r\to\infty} \sum_{v\in\mathcal{W}} v \times f_{o,r}(v) = (1,1,1,1,1).$$

Since $\mathcal{T}$ has positive density, $\liminf_{r\to\infty} f_{o,r}(v) > 0$ for every $v \in \mathcal{W}$. Let

$$f'_{o,r}(v) = \frac{1}{2}\frac{|\{s \in \mathcal{IV}_H(G_{o,r}) : V(s) = \frac{1}{2}v\}|}{|\mathcal{T}(G_{o,r})|} + \frac{|\{s \in \mathcal{IV}_F(G_{o,r}) : V(s) = v\}|}{|\mathcal{T}(G_{o,r})|}$$

Since $v \in \mathbb{N}^5$ cannot be the vector type of both a half vertex, and a full vertex in the same tiling, we have also $\liminf_{r\to\infty} f'_{o,r}(v) > 0$ for every $v \in \mathcal{W}^c$. Moreover

$$\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} v \times f'_{o,r}(v) = (1,1,1,1,1).$$

Let $u \in \mathbb{R}^5$ and $v^+ \in \mathcal{W}^c$ such that $u \cdot (1,1,1,1,1) = 0$ and $u \cdot v^+ > 0$. Suppose for the sake of contradiction that for every $v' \in \mathcal{W}^c$, $u \cdot v' \geq 0$. Then $\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} (u \cdot v) \times f'_{o,r}(v) = 0$. We have a contradiction since $\lim_{r\to\infty} \sum_{v\in\mathcal{W}^c} (u \cdot v) \times f'_{o,r}(v) \geq (u \cdot v^+) \times \liminf_{r\to\infty} f'_{o,r}(v^+) > 0$. □

By Proposition 2, if $\mathcal{T}$ has positive density, then $\mathcal{W}^c$ is good.

Compatible vectors. Let $\mathrm{span}(V)$ be the set of vectors which are linear combinaisons of vectors in V. Let

$$\mathrm{Compat}(V) = \{w \in \mathbb{N}^5 : (w,2) \in \mathrm{span}(\{(1,1,1,1,1,3)\} \cup \{(v,2) : v \in V\})\}.$$

Note that if $\mathcal{X}$ is good, then $\mathrm{Compat}(\mathcal{X})$ is also good (but the converse is not necessarily true).

Given a subset $\mathcal{X} \subseteq \mathbb{N}^5$, we define by $\mathfrak{P}_\mathcal{X}$ the subset of $\mathbb{R}^5$ such that $\alpha = (\alpha_1, \dots \alpha_5) \in \mathfrak{P}_\mathcal{X}$ if and only if for every $i \in \{1, \dots, 5\}$, $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^5 \alpha_i = 3$ and for every $v \in \mathcal{X}$, $\alpha \cdot v = 2$. One has $\mathfrak{P}_\mathcal{X} = \mathfrak{P}_{\mathrm{Compat}(\mathcal{X})}$.

If $\mathcal{T}$ is a tiling by a convex pentagon $\mathcal{P}$ of angles $(\alpha_1 \cdot \pi, \dots \alpha_5 \cdot \pi)$, then $(\alpha_1, \dots \alpha_5) \in \mathfrak{P}_{\mathcal{W}^c(T))} \cap]0,1[^5$. Moreover, if $\mathfrak{P}_\mathcal{X} \cap]0,1[^5 \neq \emptyset$, then the set $\mathrm{Compat}(\mathcal{X})$ is finite.

In the next section, we compute all good sets $\mathcal{X}$ with $\mathfrak{P}_\mathcal{X} \cap]0,1[^5 \neq \emptyset$. We show in particular that there a finitely many such sets.

3 Computation of all good subsets

We say that the permutation in S_5 is a *rotation/mirror* if it can be generated by the permutations (12345) and (3)(24)(15). Given a permutation $p \in S_5$ and a vector $v \in \mathbb{N}^5$, let $p(v)$ be the vector $(v_{p(1)}, \dots, v_{p(5)})$. Let $p(V)$ for $V \subseteq \mathbb{N}^5$, be $\{p(v) : v \in V\}$. In this section, we show the following:

Lemma 2. *If $\mathcal{X}$ is a non empty good set such that $\mathfrak{P}_\mathcal{X} \cap]0,1[^5 \neq \emptyset$ then $\mathfrak{P}_\mathcal{X} = \mathfrak{P}_{r(\mathrm{Compat}(\mathcal{B}_i))}$ for an integer $i \in \{1, \dots 371\}$ and a rotation/mirror r. ($\mathcal{B}_i$ is given in Tables 1.)*

i	$\mathcal{B}_i$	i	$\mathcal{B}_i$
~~1~~	11100	2	11010

Table 1a: $\dim(\mathfrak{P}) = 3$. Striked out numbers correspond to families of pentagons of Type 1.

i	$\mathcal{B}_i$	i	$\mathcal{B}_i$	i	$\mathcal{B}_i$
~~3~~	00003 11100	4	00003 11010	~~5~~	11100 00004
6	11010 00004	~~7~~	00012 00111	8	00012 11010
9	00012 01011	~~10~~	00012 01110	11	00012 01200
12	00012 21000	13	00012 10110	14	00012 02100
15	00012 20100	~~16~~	00012 10011	17	00012 10200
~~18~~	00102 11100	~~19~~	11010 11100	~~20~~	11001 11100
~~21~~	01002 11100	~~22~~	10110 11100	23	00102 01020
24	00102 10110	25	00102 02010	26	00102 10101
27	00102 10020	28	01011 11010		

Table 1b: $\dim(\mathfrak{P}) = 2$

i	$\mathcal{B}_i$	i	$\mathcal{B}_i$	i	$\mathcal{B}_i$
~~29~~	00003 00012 00102	~~30~~	00003 00012 11001	31	00003 00012 01002
32	00003 00012 01200	33	00003 00012 10101	34	00003 00012 02100
35	00003 00012 20100	~~36~~	00003 00102 11001	~~37~~	00003 11010 11100
~~38~~	00003 01002 10101	~~39~~	00003 10110 11100	~~40~~	00003 01110 11100
~~41~~	00003 01200 11100	~~42~~	00003 02100 11100	~~43~~	00003 10200 11100
44	00003 00102 01020	45	00003 00102 21000	46	00003 00102 02010
47	00003 02010 20100	48	00003 02100 20010	49	00003 00120 21000
50	00003 00120 10110	51	00003 00120 12000	52	00003 10110 11010
53	00003 02010 11010	54	00003 10020 11010	55	00003 00210 12000
56	00003 01200 20010	57	00003 01020 20100	58	00003 02010 10200
~~59~~	11001 11010 11100	~~60~~	10101 10110 11100	~~61~~	01200 11100 00004
~~62~~	02100 11100 00004	63	01020 11010 00004	64	02010 11010 00004
~~65~~	00111 01020 10200	~~66~~	00111 01200 10020	67	00120 01011 12000
~~68~~	00120 10011 21000	69	00210 01101 12000	70	01011 02100 10020
~~71~~	01020 10011 20100	72	01101 02010 10200	73	00012 00102 01011
~~74~~	00012 00102 01020	75	00012 00102 21000	76	00012 00102 10020
~~77~~	00012 00102 10011	78	00012 00102 12000	~~79~~	00012 00111 01002
~~80~~	00012 00111 01011	~~81~~	00012 00111 02001	82	00012 00120 21000
83	00012 00120 10002	84	00012 00120 12000	85	00012 01101 21000
86	00012 02001 20100	87	00012 10101 12000	88	00012 01002 20100
89	00012 02100 10002	90	00012 00201 01011	91	00012 00201 21000
~~92~~	00012 00201 10011	93	00012 00201 12000	~~94~~	00012 01011 10011
95	00012 10110 11010	~~96~~	00012 01110 11010	97	00012 02100 11010
98	00012 11010 20100	~~99~~	00012 01002 10011	~~100~~	00012 10200 11001
~~101~~	00012 01200 11001	102	00012 12000 20100	103	00012 01002 10200
104	00012 01020 20100	105	00012 01020 10200	106	00012 01101 20100
~~107~~	00012 01110 10110	~~108~~	00012 01110 10200	109	00012 02001 10200
110	00012 02100 10020	~~111~~	00111 01110 20100	112	00201 02010 10200
~~113~~	10011 10200 11010	~~114~~	00102 02010 11100	~~115~~	00102 11100 20010
~~116~~	01020 01101 11100	~~117~~	00201 01020 11100	118	00201 02010 20100
119	02001 10200 11010	120	00102 02001 10020		

Table 1c: $\dim(\mathfrak{P}) = 1$

i	$\mathcal{B}_i$	i	$\mathcal{B}_i$	i	$\mathcal{B}_i$
~~121~~	00003 00012 00102 01002	~~122~~	00003 00012 00102 04000	~~123~~	00003 00012 02100 11001
~~124~~	00003 00012 11001 20100	125	00003 00012 20100 21000	126	00003 00012 02100 12000
127	00003 00012 02100 21000	128	00003 00012 12000 20100	129	00003 00012 01002 00400
130	00003 00012 10101 21000	131	00003 00012 02100 00400	132	00003 00012 20100 00400
133	00003 00012 21000 03100	134	00003 00012 12000 30100	135	00003 00012 01200 20100
136	00003 00012 01200 04000	137	00003 00012 20100 01300	~~138~~	00003 00102 02010 11001
~~139~~	00003 02010 02100 11010	~~140~~	00003 00210 01002 10101	~~141~~	00003 00210 01200 10110
~~142~~	00003 00210 01110 10200	~~143~~	00003 01002 10101 20010	144	00003 00102 20010 21000
~~145~~	00003 00111 01200 20010	~~146~~	00003 00111 02010 10200	147	00003 01011 02100 20010
~~148~~	00003 02010 10011 20100	149	00003 01002 20010 20100	150	00003 00210 01011 12000
~~151~~	00003 00210 10011 21000	152	00003 00210 02100 00031	153	00003 00210 20100 00031
154	00003 02100 21000 00031	155	00003 12000 20100 00031	156	00003 12000 20010 00031
157	00003 02010 21000 00031	158	00003 10200 20010 00031	159	00003 01200 02010 00031
160	00003 01200 20100 00031	161	00003 01200 12000 00031	162	00003 02100 10200 00031
163	00003 10200 21000 00031	164	00003 00102 02010 21000	165	00003 01101 02010 20100
166	00003 02100 10101 20010	167	00003 00210 01002 20100	168	00003 00210 01101 21000
169	00003 01002 10200 20010	~~170~~	00003 01200 11001 20010	171	00003 00102 02010 00040
172	00003 02010 20100 00040	173	00003 02100 20010 00040	174	00003 00210 01002 00040
175	00003 00210 21000 00040	176	00003 00210 02100 00040	177	00003 00210 12000 00040
178	00003 00210 20100 00040	179	00003 01002 20010 00040	180	00003 12000 20100 00040
181	00003 02100 21000 00040	182	00003 01200 20010 00040	183	00003 01200 02010 00040
184	00003 01200 20100 00040	185	00003 12000 20010 00040	186	00003 02010 10200 00040
187	00003 02010 21000 00040	188	00003 10200 21000 00040	189	00003 10200 20010 00040
190	00003 02010 20100 00400	191	00003 02100 20010 00400	192	00003 00210 21000 04000
193	00003 01200 20010 04000	194	00003 00102 21000 03010	195	00003 02010 20100 01030
196	00003 02100 20010 10030	197	00003 01002 20100 00310	198	00003 01101 02010 21000
199	00003 10101 12000 20010	200	00003 00210 21000 00130	201	00003 00210 01101 20100
202	00003 00210 02100 10101	203	00003 00210 12000 00130	~~204~~	00003 10200 11001 20010
~~205~~	00003 01200 02010 11001	206	00003 01002 10200 30010	207	00003 01200 20010 10030
208	00003 02010 10200 01030	209	00003 02100 20010 01021	210	00003 02010 20100 10021
211	00003 01200 20010 00121	212	00003 00210 21000 10021	213	00003 01200 02010 20100
214	00003 02100 10200 20010	215	00003 00210 02100 21000	216	00003 00210 12000 20100
217	00003 01200 12000 20010	218	00003 02010 10200 21000	219	00003 02010 20100 01300
220	00003 02100 20010 10300	221	00003 00210 21000 03100	222	00003 00210 12000 30100
223	00003 01200 20010 13000	224	00003 02010 10200 31000	~~225~~	00003 00111 01020 20100
~~226~~	00003 00111 01020 20010	~~227~~	00003 00111 01200 20100	228	00003 01020 02100 20010
229	00003 02010 10020 20100	230	00003 02100 20010 01030	231	00003 02010 20100 10030
232	00003 00120 01020 10110	233	00003 00120 01011 21000	234	00003 00120 20010 00301
235	00003 00120 01011 20010	236	00003 00120 21000 00310	~~237~~	00003 00120 02010 10011
238	00003 00120 02010 21000	239	00003 00120 12000 20010	240	00003 00120 21000 00400
241	00003 00120 02010 00400	242	00003 00120 20010 00400	243	00003 00120 21000 03010
244	00003 00120 12000 30010	245	00003 00120 01200 20010	246	00003 00120 01200 10101
247	00003 00120 01200 04000	248	00003 00120 20010 01300	249	00003 00120 10200 03001
~~250~~	00003 10011 12000 20100	251	00003 12000 20010 00130	252	00003 01200 20010 00130
253	00003 00210 01020 20100	254	00003 00210 01020 04000	255	00003 00210 20100 01030
256	00003 01020 20100 00211	257	00003 01020 20010 00301	258	00003 10020 20100 00301
259	00003 02010 10020 00301	260	00003 01020 20100 00310	261	00003 01020 20100 04000
262	00003 01020 02100 00400	263	00003 10020 20100 00400	~~264~~	02001 02010 02100 11001
~~265~~	00201 00210 01200 10101	~~266~~	00021 02010 11001 20100	~~267~~	00021 02100 11001 20010
268	00021 00210 10101 21000	269	00021 00210 01101 12000	270	00021 00210 02100 00004
271	00021 00210 20100 00004	272	00021 02100 21000 00004	273	00021 12000 20100 00004
274	00021 01200 10101 20010	275	00021 01101 02010 10200	276	00021 01200 02010 00004
277	00021 01200 20100 00004	278	00021 01200 12000 00004	279	00021 02010 21000 00004
280	00021 10200 21000 00004	281	00021 12000 20010 00004	282	00021 02100 10200 00004
283	00021 10200 20010 00004	~~284~~	00201 02010 11001 20100	~~285~~	00201 02100 11001 20010
286	00210 02001 10101 21000	287	01200 02001 10101 20010	288	00210 02001 20100 00004
289	00201 02010 21000 00004	290	00201 12000 20010 00004	291	02001 10200 20010 00004
~~292~~	02100 10200 11001 20010	~~293~~	01200 02010 11001 20100	294	01200 10101 12000 20010
295	01101 02010 10200 21000	296	00210 02100 10101 21000	297	00210 01101 12000 20100
~~298~~	00111 01020 10200 20010	~~299~~	00111 01200 10020 20100	~~300~~	00120 00201 10011 21000
301	00120 00201 02010 00004	302	00120 00201 20010 00004	303	00120 02001 20010 00004
304	00120 01011 12000 20010	~~305~~	00120 02010 10011 21000	306	00120 01200 10101 20010
307	00120 01200 02001 00004	~~308~~	01020 02001 10011 20100	309	02001 10020 20100 00004
310	01200 02001 20100 00004	311	00201 12000 20100 00004	312	00201 10020 20100 00004
~~313~~	00210 01020 10011 20100	~~314~~	00021 00111 01020 20010	~~315~~	00021 00111 02010 10020
~~316~~	00021 00120 01011 20010	317	00021 00210 01011 10020	~~318~~	00021 00210 01110 20100
319	00021 00210 02100 10110	320	00021 01200 02010 10200	~~321~~	00120 00201 01110 20010
322	00120 00201 02010 10110	~~323~~	01020 01110 02001 20010	324	00021 01200 02010 20100
325	00021 02100 10200 20010	326	00021 00210 02100 21000	327	00021 00210 12000 20100
328	00021 01200 12000 20010	329	00021 02010 10200 21000	330	00201 01020 02100 20010

Table 1d: $\dim(\mathfrak{P}) = 0$ (part 1/2)

i	$\mathcal{B}_i$	i	$\mathcal{B}_i$	i	$\mathcal{B}_i$
331	00201 02010 10020 20100	332	00120 01200 02001 20010	333	00210 02001 10020 21000
334	00120 00201 02010 21000	335	00210 01020 02001 20100	336	00012 00120 21000 00301
337	00012 00120 12000 00301	338	00012 00120 21000 03001	339	00012 00120 12000 30001
340	00012 00120 01200 13000	341	00012 00120 02001 10300	342	00012 00120 10200 03100
343	00012 02001 20100 01030	344	00012 02001 20100 01300	345	00012 00201 21000 00130
346	00012 00201 12000 00130	347	00012 02001 21000 00130	348	00012 10200 21000 00130
349	00012 00201 12000 30100	350	00012 02100 10200 01120	351	00012 12000 20100 01300
352	00012 00201 01020 13000	353	00012 00201 20100 01030	354	00012 00201 02100 10030
355	00012 00201 10020 03010	356	00012 01020 20100 00211	357	00012 01020 20100 00301
358	00012 02100 10020 00301	359	00012 02001 10020 00310	360	00012 12000 20100 01030
361	00012 01020 12000 01300	362	00012 01020 10200 03001	363	00012 01020 10200 30001
364	00012 02001 10200 01030	365	00012 02001 10200 31000	366	00012 10020 20100 01300
367	01020 02001 20100 00013	368	02001 10020 20100 00013	369	01020 02001 10200 00013
370	01020 10200 20010 00013	371	01020 02001 10200 30010		

Table 1e: $\dim(\mathfrak{P}) = 0$ (part 2/2)

The remaining of this section is devoted to the proof of Lemma 2, which is algorithmic. In this section, the order of the angles is not important, so we suppose w.l.o.g. that there is an $\alpha = (\alpha_1, \ldots, \alpha_5) \in \mathfrak{P}_{\mathcal{X}}$ such that $1 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 > 0$.

Let $\mathfrak{P}^{\geq}_{\mathcal{X}}$ be the set of vectors $(\alpha_1, \ldots \alpha_5)$ such that $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$, $\sum_i \alpha_i = 3$ and for every $v \in \mathcal{X}$, $v \cdot \alpha = 2$. Clearly, $\mathfrak{P}^{\geq}_{\mathcal{X}}$ is a convex polytope. Clearly, one has $\mathfrak{P}^{\geq}_{\mathcal{X}} = \mathfrak{P}_{\mathcal{X}} \cap \mathfrak{P}^{\geq}_{\emptyset}$ and $\mathfrak{P}^{\geq}_{\mathcal{X}} = \mathfrak{P}^{\geq}_{\text{Compat}(\mathcal{X})}$. If $\mathfrak{P}^{\geq}_{\mathcal{X}}$ is non empty, let $m_{\mathcal{X}} \in [0,1]^5$ be such that $(m_{\mathcal{X}})_i = \min\{\alpha_i : \alpha \in \mathfrak{P}^{\geq}_{\mathcal{X}}\}$. Note that $(m_{\mathcal{X}})_1 \geq \frac{3}{5}$, $(m_{\mathcal{X}})_2 \geq \frac{1}{2}$, $(m_{\mathcal{X}})_3 \geq \frac{1}{3}$, and $(m_{\mathcal{X}})_i \geq (m_{\mathcal{X}})_{i+1}$ for every $i \in \{1, \ldots 4\}$.

We say that a set $\mathcal{X}$ is *maximal* if $\mathcal{X} = \text{Compat}(\mathcal{X})$. To prove Lemma 2, it suffices to prove it for every maximal good set.

The procedure Recurse (Algorithm 1) computes all maximal good sets $\mathcal{Y} \supseteq \mathcal{X}$ with $\mathfrak{P}^{\geq}_{\mathcal{Y}} \cap]0,1[^5 \neq \emptyset$.

We know that for all maximal good set $\mathcal{Y} \supseteq \mathcal{X}$, one has $\text{Compat}(\mathcal{X}) \subseteq \mathcal{Y}$ (line 2). Since $u \cdot (1,1,1,1,1) = 0$, for every $v \in \mathcal{X}$, $v \cdot u = 0$, and by definition of good subsets, we know that if there is another maximal good set $\mathcal{Y} \supsetneq \mathcal{X}$, then there is a $w \in \mathbb{N}^5 \setminus \mathcal{X}$ such that $w \cdot u \geq 0$. Moreover, we must have $w \cdot m_{\mathcal{X}} \leq 2$, otherwise $\mathfrak{P}_{\mathcal{Y}} \cap]0,1[^5$ would be empty. Thus, w is in the set V computed line 8. We try every possibility for w at line 10. An important point of the algorithm is that V is finite: if $v \in V$, then for every i such that $(m_{\mathcal{X}})_i > 0$, v_i is bounded by $\frac{2}{m_{\mathcal{X}}}$, and thus for every i with $(m_{\mathcal{X}})_i = 0$, v_i is bounded by $-\frac{1}{u_i} \sum_{j:(m_{\mathcal{X}})_j > 0} (v_j \max(0, u_j))$.

The computation of a u (line 7) with the required property is done using a linear program. If no such u exists, then the algorithm would fail. But, even it is not necessary for the proof (it suffices that one possible execution terminates), one can show that this case never happens.

Algorithm 1 Exhaustive search

1: **procedure** RECURSE($\mathcal{X}$)
2: $\quad\mathcal{X} \leftarrow \text{Compat}(\mathcal{X})$
3: $\quad$**if** $\mathfrak{P}_{\mathcal{X}}^{\geq} \cap]0,1[^5 = \emptyset$ **then return end if**
4: $\quad$**if** $\mathcal{X}$ is good **then**
5: $\quad\quad$Add $\mathcal{X}$ to the list of good sets
6: $\quad$**end if**
7: $\quad$Let $u \in \mathbb{R}^5$ such that:
- $u \cdot (1,1,1,1,1) = 0$
- $\forall v \in \mathcal{X}$, $u \cdot v = 0$ and
- $\forall i \in \{4,5\}$, $(m_{\mathcal{X}})_i = 0 \Rightarrow u_i < 0$

8: $\quad V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \text{ and } v \cdot m_{\mathcal{X}} \leq 2\}$
9: $\quad$**for** every $w \in V \setminus \mathcal{X}$ **do**
10: $\quad\quad$RECURSE($\mathcal{X} \cup \{w\}$)
11: $\quad$**end for**
12: **end procedure**

Proposition 3. *In line 7, such a u always exists.*

Proof. Note that if $\alpha, \alpha' \in \mathfrak{P}_{\mathcal{X}}$, then $(\alpha - \alpha') \cdot (1,1,1,1,1) = 0$ and for every $v \in \mathcal{X}$, $(\alpha - \alpha') \cdot v = 0$. If $(m_{\mathcal{X}})_4 > 0$ and $(m_{\mathcal{X}})_5 > 0$, one can take $u = (0,0,0,0,0)$. If $(m_{\mathcal{X}})_4 > 0$ and $(m_{\mathcal{X}})_5 = 0$, there is $\alpha \in \mathfrak{P}_{\mathcal{X}}^{\geq}$ such that $\alpha_5 = 0$. Otherwise $(m_{\mathcal{X}})_4 = (m_{\mathcal{X}})_5 = 0$, and there is $\alpha \in \mathfrak{P}_{\mathcal{X}}^{\geq}$ such that $\alpha_4 = \alpha_5 = 0$. In all cases, $u = \alpha - \alpha'$, with $\alpha' \in \mathfrak{P}_{\mathcal{X}}^{\geq} \cap]0,1[^5 \neq \emptyset$, has the desired properties. □

The procedure RECURSE is non deterministic, and a good choice for u can reduce the size of the research tree and the computation time. But, since the dimension of the subspace spanned by $\mathcal{X}$ strictly increase at every recursive call, there is at most 5 nested calls of RECURSE, and this procedure always terminates.

Computation. In order to reduce the computation time, we also track vectors w which are not in $\mathcal{Y}$. Our implementation takes approximately 40 seconds to explore all the cases (1354 calls of RECURSE). There are 193 non-empty maximal goods sets $\mathcal{X}$ with $\mathfrak{P}_{\mathcal{X}}^{\geq} \cap]0,1[^5 \neq \emptyset$, and (taking all permutations) 3495 sets with $\mathfrak{P}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$. If we keep only one representative for each class up to rotation/mirror, one has the 371 sets of Tables 1.

4 Testing all 371 cases

For each of the 371 cases, we do an exhaustive search by backtracking, to try to construct a tiling of an arbitrarily large region. If this backtracking is finite, then we know that there is no pentagon with these angles condition which tiles the plane.

Throughout this section, one fix $\mathcal{X} = \text{Compat}(\mathcal{B}_i)$ for an $i \in \{1, \dots 371\}$. One supposes that $\mathcal{P}$ pentagon which tiles the plane with a tiling $\mathcal{T}$ of positive density, such that $\text{Compat}(\mathcal{W}^c(\mathcal{T})) = \mathcal{X}$. Let $\mathfrak{P} = \mathfrak{P}_{\mathcal{X}}$.

Suppose that the vertices of $\mathcal{P}$ are s_i (with angle $\alpha_i \times \pi$) and the lengths of the sides are ℓ_i, $i \in \{1, \dots 5\}$, in clockwise order, and such that ℓ_1 is the length between s_1 and s_2. Moreover, we suppose w.l.o.g. than $\sum_i \ell_i = 1$.

The backtracking is done on two data-structures: a tiling graph which represents the geometric information we have for the part of the tiling, and a linear program Q with represent conditions we have on ℓ.

Tiling graph. The *tiling graph* is an embedded planar graph, with additional information (label on angles, edge and faces). (Note that this graph differs significantly from the graph defined in Section 2.)

Each vertex of the graph corresponds to a vertex of the tiling. (This mapping is not necessarily injective.) Each angle has a type: either 1, 2, 3, 4, 5, $\emptyset$, π or ?. Each edge in the graph has also a type: either 1, 2, 3, 4 or 5. The planar graph has two types of faces. A face is either a *normal* face, or is a *special* face. Each edge is adjacent to one special face, and one normal face.

There is a bijection between the normal faces and the tiles of the tiling. Thus a normal face has degree is 5, and the types of its angles and edges are either (in clockwise order) 1,2,3,4,5 or 5,4,3,2,1. Moreover the type of the edge between the angles 1 and 2 is 1.

A special face corresponds either to the frontier between tiles, or an unknown area of the plane. Its angles are $\emptyset$, π or ?. An angle of type $\emptyset$ (resp. π) corresponds to an angle of 0 (resp. π) in the tiling. A special face is *complete* if it has no ? angles. A complete special face has exactly two $\emptyset$ angles. In this case, it corresponds to a segment which is a frontier between two or more tiles.

A vertex v is *complete* if there is no ? angle adjacent to it. Similarly to Section 2, let $V^c(v) \in \mathbb{N}^5$ be such that, for every $i \in \{1, \dots 5\}$, $(V^c(v))_i = c \times (V(v))_i$, where $(V(v))_i$ is the number of angles i adjacent to v, and $c = 1$

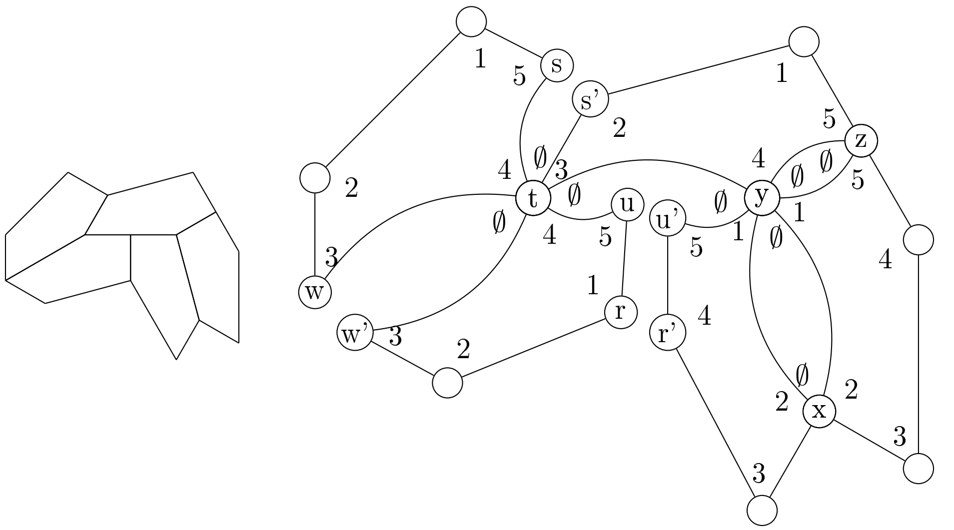

Figure 1: Example of a tiling graph (Type 15). Unmarked angles are labeled "?"

if there is no π adjacent to v (*i.e.* v is full) and $c = 2$ if there is one angle π adjacent to v (*i.e.* v is half).

An example of a tiling graph is given in Figure 1. The tiling graph the right correspond to the tiling on the left. Note that other tiling graphs are possible to represent the same tiling.

A *run* on a special face is a succession of consecutive $\emptyset$ and π angles. Each run corresponds to aligned points in the tiling. For example, on Figure 1, (s, t, s'), (u, t, y, u') are (maximal) runs. The complete face (y, z) induces also a run.

Since we want to generate a tiling graph corresponding to a tiling by $\mathcal{P}$, we keep the following conditions on the tiling graph (that is, we backtrack if one of these conditions are not fulfilled)

- for every vertex v, there is a $w \in \mathcal{X}$ such that $V^c(v) \leq w$,
- for every complete vertex v, $V^c(v) \in \mathcal{X}$,
- there is no run with more than two $\emptyset$ angles,
- there is no vertex with two π angles adjacent to it.

Note that every finite sub-set $\mathcal{T}'$ of $\mathcal{T}$ can ge represented by a tiling graph with the previous properties (but the representation is not unique). We will make some "completion" operations on it, which guarantee that the new tiling graph is also a tiling graph of the same tile set.

Moreover, during the exploration, all the operations we do on the tiling graph keep the additional following conditions: the graph is connected, has exactly one non-complete face, and has no vertices with most than one ? angles adjacent to it.

Completing vertices. At every time, as soon as there is a non-complete vertex v such that $V^c(v) \in \mathcal{W}^c$, we relabel the angle labeled ? adjacent to v with the label $\emptyset$. Moreover, for every non-complete vertex v such that $2 \times V(v) \in \mathcal{W}^c$, we relabel the angle labeled ? adjacent to v with the label π.

Length suppositions and completing faces. If there is a pair of vertices (v, v') on a same run, and the linear program Q imply that v and v' are the same point in the tiling, then we merge v and v' in the graph.

If Q does not permit to decide among the following 3 possibilities :

- v and v' are the same point in the tiling,
- v is on the right of v' (with an arbitrary orientation of the line corresponding to the run),
- v is on the left of v,

then we branch on the 3 possibilities: we add the corresponding condition on Q, and recurse.

Example. In Figure 1, (w, t, w') is a run, and the length between t and w (which is ℓ_3) is the same as the distance between t and w'. Thus we merge w and w', and we relabel the angle t, w, t into $\emptyset$. We created a new special complete face (t, w).

We have also to consider the run (u, t, y, u'). We have either to choose if u and u' is the same point (that is, add the condition $\ell_3 = \ell_4 + \ell_5$), or not (and in this case, explore with the additional condition $\ell_3 > \ell_4 + \ell_5$).

Suppose now we explore the first case (merge of u and u'). We create an angle π adjacent to u to complete the special face (u, t, y). Since $2\alpha_5 = \pi$ in the type 15 (i=303), the vertex u is now complete, and we can label the angle r, u, r' as $\emptyset$. We have a new run (r, u, r'). Since we have already the condition $\ell_4 = \ell_5$ in Q (by the complete special face y, z), we know that r and r' must be the same vertex in the tiling, and we can merge r and r' without branching.

Existence of a solution which respects Q Let $s(\alpha)$ be the vector such that $s(\alpha)_i = (i-1) - \sum_{j=1}^{i-1} \alpha_i$. Note that if α (resp. ℓ) is the vector of angles (resp. lengths) of a pentagon, we have

$$\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0. \qquad (1)$$

Given a linear program Q, we denote by $\mathfrak{Q}$ the set of $\ell \in \mathbb{R}^5$ such that $\ell \geq 0$, $\sum \ell = 1$ and ℓ respects the conditions in Q.

If the following condition is not fulfilled, then no convex pentagon exists with the conditions, and one backtrack.

$$\exists \ell \in \mathfrak{Q} \cap]0,1[^5, \exists \alpha \in \mathfrak{P} \cap]0,1[^5, \sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0 \qquad (2)$$

If $\dim(\mathfrak{P}) = 0$ (cases from 121 to 371), since all conditions for $\mathfrak{P}$ are rational, there is a $p \in \mathbb{Z}^5$ and $q \in \mathbb{N}^+$ such that $s(\alpha) = p/q$, where $\{\alpha\} = \mathfrak{P}$, and thus the condition $\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0$ can be turned into $\ell \cdot \cos = 0$ and $\ell \cdot \sin = 0$, where $\cos_i = \cos(p_i \times \pi/q)$ and $\sin_i = \sin(p_i \times \pi/q)$. The one can decide with computations on an algebraic extension of $\mathbb{Q}$ (for example $\mathbb{Q}[\cos(\pi/q)]$).

If $\dim(\mathfrak{P}) > 0$, the verification of (2) is more complicated, and we do not try to verify it at each recursion. However, if one has a certificate that (2) is false, then we backtrack. If there is a family of polytopes $\mathfrak{P}_l$, $l \in \{1, \ldots, L\}$ such that $\mathfrak{P} \subseteq \bigcup_{l=1}^{L} \mathfrak{P}_l$, and for every $l \in \{1, \ldots, L\}$, $\{x \in \mathfrak{Q} : x \cdot \sin^+ \geq 0, x \cdot \sin^- \leq 0, x \cdot \cos^+ \geq 0 \text{ and } x \cdot \cos^- \leq 0\} = \emptyset$, where $\sin_i^+$ (resp. $\sin_i^-$) is an upper (resp. lower) bound of $\{\sin(\pi s(\alpha)_i) : \alpha \in \mathfrak{P}_l\}$ (and similarly for $\cos^+$ and $\cos^-$), then we know that (2) if false. This can be done using rational numbers.

This procedure cannot decide, for example, if (2) is false, but (1) has a degenerate solution (ℓ, α) is on the boundary of $\mathfrak{Q} \times \mathfrak{P}$. To resolve these cases, we also backtrack in some degenerate solutions (types 20 to 24 in Table 2).

Branching. If we are not in any case of backtracking, then we add a new normal face to the tiling graph.

We take a non-complete vertex w in the graph. We know that, if the tiling graph corresponds to a sub-tiling $\mathcal{T}'$ of a tiling $\mathcal{T}$ by $\mathcal{P}$, there is a tile $P \in \mathcal{T} \setminus \mathcal{T}'$ such that w is a vertex of P, and P shares a line segment with $\mathcal{T}'$. Then we branch on on all theses possibilities of face addition.

Type 1 (i=1)	$a+b+c=2\pi$		Type 2 (i=2)	$a+b+d=2\pi$	$C=E$
Type 3 (i=31)	$3e=2\pi$ $d+2e=2\pi$ $b+2e=2\pi$	$C+E=D$ $A=B$	Type 4 (i=6)	$a+b+d=2\pi$ $2e=\pi$	$D=E$ $B=C$
Type 5 (i=4)	$3e=2\pi$ $a+b+d=2\pi$	$D=E$ $B=C$	Type 6 (i=13)	$d+2e=2\pi$ $a+c+d=2\pi$	$C=D=E$ $A=B$
Type 7 (i=17)	$d+2e=2\pi$ $a+2c=2\pi$	$A=C=D=E$	Type 8 (i=14)	$d+2e=2\pi$ $2b+c=2\pi$	$A=B=C=D$
Type 9 (i=15)	$d+2e=2\pi$ $2a+c=2\pi$	$A=B=C=D$	Type 10 (i=69)	$2c+d=2\pi$ $b+c+e=2\pi$ $a+2b=2\pi$	$A+C=D=E$
Type 11 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A=B=C+2E$	Type 12 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A+C=B=2E$
Type 13 (i=63)	$b+2d=2\pi$ $a+b+d=2\pi$ $2e=\pi$	$A=2B=2C$	Type 14 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A=B=2C=$ $2E$
Type 15 (i=303)	$c+2d=2\pi$ $2b+e=2\pi$ $2a+d=2\pi$ $2e=\pi$	$B=D=E$ $C=2B$	Type 16 (i=72) $\subset$ T10	$b+c+e=2\pi$ $2b+d=2\pi$ $a+2c=2\pi$	$2A=D=E$ $A=C$
Type 17 (i=25) $\subset$ T2	$c+2e=2\pi$ $2b+d=2\pi$	$A=B=C=$ $D=E$	Type 18 (i=73) $\subset$ T2	$d+2e=2\pi$ $c+2e=2\pi$ $b+d+e=2\pi$	$D=E$ $A=B$
Type 19 (i=23) $\subset$ T1	$c+2e=2\pi$ $b+2d=2\pi$	$A=B=C=D$	Type 20 (i=2) degen.	$a+b+d=2\pi$	$A=C+D$ $B=E$
Type 21 (i=12) degen.	$d+2e=2\pi$ $2a+b=2\pi$	$A=B$ $C=D$	Type 22 (i=27) degen.	$c+2e=2\pi$ $a+2d=2\pi$	$A=B=C=E$
Type 23 (i=64) degen.	$2b+d=2\pi$ $a+b+d=2\pi$ $2e=\pi$	$A=2C=2D$	Type 24 (i=69) degen.	$2c+d=2\pi$ $b+c+e=2\pi$ $a+2b=2\pi$	$2D=A+C$ $2E=A+C$

Table 2: Conditions for tilings of types 1 to 24, with $(a,\dots,e)=(\alpha_1,\dots,\alpha_5)$ and $(A,\dots,E)=(\ell_1,\dots,\ell_5)$.

Results. If we also backtrack if we are in one the 24 types presented in Table 2, the exhaustive search terminates, for all 371 cases for angle conditions. That is, if a convex pentagon $\mathcal{P}$ tiles the plane, then $\mathcal{P}$ is in one the 24 families.

Types 1 to 15 are the already known families of pentagons which tiles the plane. Types 16 to 19 are special cases of the 15 already known families, *i.e.* every non-degenerate solution of (2) is in a known family. Type 16 is a special case of Type 10: the only solution of (2) has $\alpha_1 = \alpha_4 = \alpha_5 = \frac{\pi}{2}$ and $\alpha_2 = \alpha_3 = \frac{3\pi}{4}$. Types 17 and 18 are special cases of Type 2: in every solution, all lengths are equal and $\alpha_1 + \alpha_3 = \pi$. Type 19 is a special case of Type 1: conditions imply that for every solution, $\alpha_3 + \alpha_4 = \pi$. Finally, types 20 to 24 are degenerate, *i.e.* (2) has no solution. These observations can be done using a computer algebra system, turning linear conditions on angles, lengths and (1) into a system of polynomial equations.

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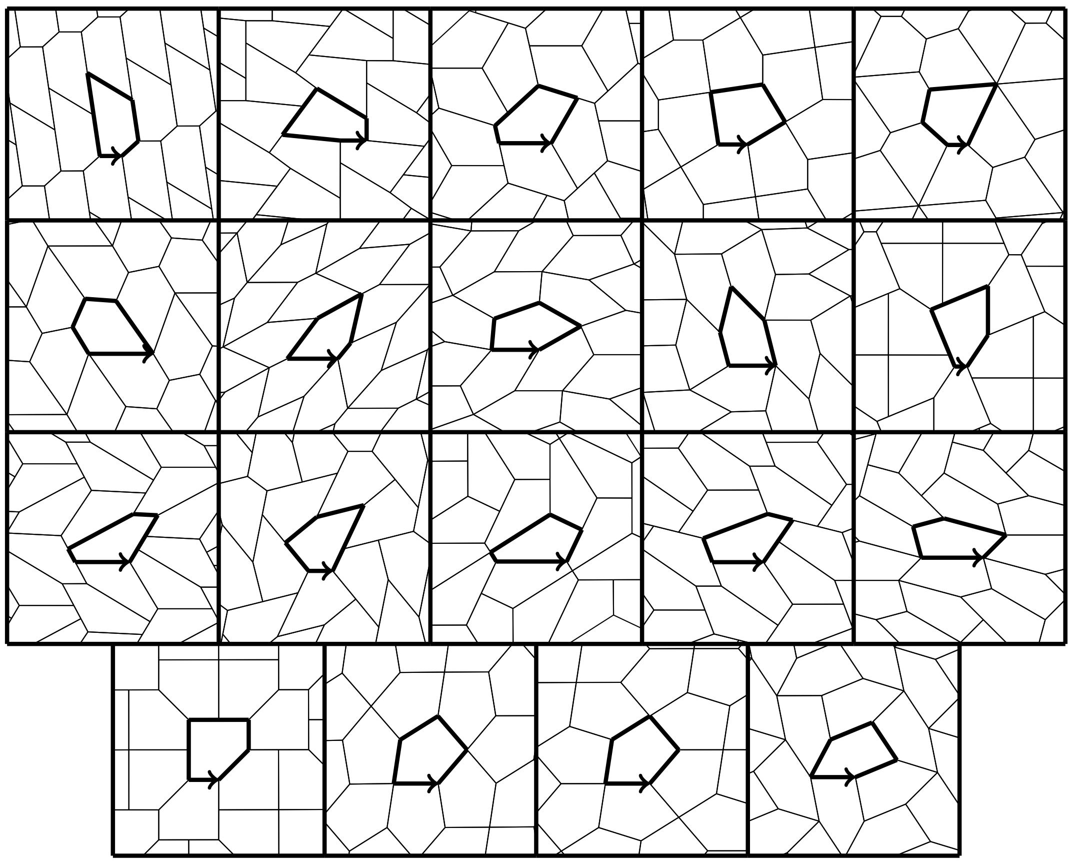

Figure 2: Tilings 1 to 19. The arrow is from vertex s_1 to s_2 in the bold tile.